(b) For some integer $n \geq 1$, let f have a continuous n^{th} derivative in the open interval (a, b). Suppose also that for some interior point c in (a, b) we have $f'(c) = f''(c) = ... = f^{(n-1)}(c) = 0$, but $f^{(n)}(c) \neq 0$. Then for n even, f has a local minimum at c if $f^{(n)}(c) > 0$, and a local maximum at c if $f^{n}(c) < 0$. If n is odd, there is neither a local maximum nor a local minimum at c.

SECTION C — (3 × 10 = 30 marks)

Answer any THREE questions.

16. Let c be a constant and f and g be two measurable functions defined on the same domain. Then the functions $f + c$, cf, $f + g$, $f - g$ and fg are also measurable.

17. State and prove Fatou's lemma.

18. State and prove Weirestrass Approximation theorem.

19. Show that if both partial derivatives $D_r f$ and $D_k f$ exist in an n-ball $B(c ; \delta)$ and if both are differentiable at c, then $D_{r,k} f(c) = D_{k,r} f(c)$.

20. State and prove the inverse function theorem.

NOVEMBER/DECEMBER 2024

23PMA22 — REAL ANALYSIS — II

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Time : Three hours Maximum : 75 marks

SECTION A — (10 × 2 = 20 marks)

Answer ALL questions.

1. Define Lebesgue outer measure.

2. Define Borel measurable.

3. Define Integrable of a measurable function.

4. Define bounded measurable functions.

5. Define orthogonal system of functions.

6. Write Dirichlet Integrals.

7. Define linear functions.

8. Write Taylor's formula of first-order derivative.

9. State Jacobian Determinant.

10. Write Cauchy-Riemann Equations.

SECTION B — (5 × 5 = 25 marks)

Answer ALL questions.

11. (a) If $m^*E = 0$, then show that E is measurable.

Or

(b) Prove that the interval (a, ∞) is measurable.

12. (a) Let ϕ and ψ be simple functions which vanish outside a set of finite measures. Then $\int(a\phi + b\psi) = a\int\phi + b\int\psi$, and if $\phi \geq \psi$ a.e. then $\int\phi \geq \int\psi$.

Or

(b) If f and g are non negative measurable functions, then

(i) $\int_E cf = c\int_E f,\ c > 0$

(ii) $\int_E f + g = \int_E f + \int_E g$

(iii) If $f \leq g$ a.e., then $\int_E f \leq \int_E g$.

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13. (a) Assume that $g(0^+)$ exists and suppose that for some $\delta > 0$, the Lebesgue integral $\int_0^\delta \frac{g(+) - g(0^+)}{t} dt$ exists. Then we have $\lim_{\alpha \to -\infty} \frac{2}{\pi}\int_0^\delta g(t)\frac{\sin \alpha t}{t} dt = g(0^+)$.

Or

(b) Assume that $f \in L(I)$. Then, for each real β, we have $\lim_{a \to +\infty} \int_t f(t)\sin(\alpha t + \beta) dt = 0$.

14. (a) If f is differentiable at c, then f is continuous at c.

Or

(b) Let S be an open connected subset of R^n, and let $f : S \to R^m$ be differentiable at each point of s. If $f'(c) = 0$ for each c in s, then f is constant on s.

15. (a) Let A be an open subset of R^n and assume that $f : A \to R^n$ is continuous and has finite partial derivatives D_{jfi} on A. If f is one-to-one on A and if $J_f(\alpha) \neq 0$ for each X in A, then $f(A)$ is open.

Or